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ISOMETRIC EXTENSIONS AND MULTIPLE RECURRENCE OF INFINITE MEASURE PRESERVING SYSTEMS

BY

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ABSTRACT

Furstenberg's multiple recurrence theorem is investigated for infinite measure preserving systems arising from isometric extensions which are 1-factor maps. It will be shown that any isometric extension preserves d-recurrence for all $d \geq 1$ and multiple recurrence.

1. Introduction

Let $\mathbf{X} = (X, \mu, T)$ be a measure preserving system and $d \geq 1$ be an integer. We say $\mathbf{X}$ is **d-recurrent** if for an arbitrary measurable subset A of X of positive measure, there exists an integer $k \geq 1$ such that

$$\mu(A \cap T^{-k}A \cap T^{-2k}A \cap \cdots \cap T^{-dk}A) > 0.$$

In other words, on the set A, $T^n x$ return back almost surely to A along the times n in an arithmetic progression of length $d+1$. We say $\mathbf{X}$ is **multiply recurrent** if $\mathbf{X}$ is d-recurrent for all $d \geq 1$.

In his beautiful theory of multiple recurrence, Furstenberg ([Fr]) showed that every finite measure preserving system is an SZ-system, which means that for any $d \geq 1$ and any measurable set A of X of positive measure,

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{-n}A \cap T^{-2n}A \cap \cdots \cap T^{-dn}A) > 0.$$

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We call it the SZ-property. This immediately implies multiple recurrence. He also showed that the SZ-property is equivalent to Szemerédi's Theorem ([Sz]) in number theory, which says that any set of positive integers with positive upper density contains arbitrarily long arithmetic progressions.

In his proof of the multiple recurrence theorem, Furstenberg introduced some classes of extensions $\mathbf{X} \to \mathbf{Y}$ of finite measure preserving systems which play a vital role in the theorem. As a matter of fact, the SZ-property is preserved by taking a primitive extension $\mathbf{X} \to \mathbf{Y}$ which is a combination of a compact extension and a weak mixing extension (Proposition 7.12 [Fr]). He obtained the structure theorem of any finite measure preserving system which says it is realized by a possibly transfinite succession of primitive extensions starting from the trivial 1-point system (Theorem 6.16 [Fr]). Thus any finite measure preserving system has the SZ-property.

When we take infinite measure preserving systems, the situation is different. Eigen–Hajian–Halverson ([Ei-Haj-Hal]) and Aaronson–Nakada ([A-N]) showed that not all infinite measure preserving systems are multiply recurrent. Furthermore, Aaronson–Nakada also showed a criterion for a conservative ergodic Markov system to be d-recurrent. As is well known, any ergodic infinite measure preserving system is of either zero type or positive type. When the system is of zero type, it can no longer be an SZ-system. Then in the infinite measure case, multiple recurrence (and d-recurrence) should be investigated rather than the SZ-system. Due to the Aaronson–Nakada's criterion we know that the Kakutani–Parry Markov shift ([Kak-P]) of infinite ergodic index, which is known to be of zero type, is multiply recurrent.

In this paper we are concerned with multiple recurrence and d-recurrence of infinite measure preserving systems. Being different from approaches of Eigen–Hajian–Halverson ([Ei-Haj-Hal]) and Aaronson–Nakada ([A-N]), we investigate extensions of infinite measure preserving systems to see lifting property of multiple recurrence and d-recurrence through extension.

In his book ([A]), Aaronson has unveiled interesting phenomena of infinite measure preserving systems, including recurrence behavior, ergodic theorems and so on. The **c-factor map** is also discussed in it and is defined as follows. If an extension α: $(X, \mu, T) \to (Y, \nu, S)$ satisfies, for $c \in (0, \infty]$, $\mu \circ \alpha^{-1}(A) = c\nu(A)$ for all measurable sets A of X, then α is called a c-factor map.

It says in [A] that a 1-factor map, which is also called measure preserving extension, acts as a factor between finite measure preserving systems. It has been conjectured (page 291, Remark [A-N]) that any measure preserving extension preserves d-recurrence for all $d \geq 1$ and multiple recurrence. We will show a partial answer of this conjecture. Namely any isometric extension, which is defined to be a factor of a compact group extension and hence a 1-factor map, preserves multiple recurrence and d-recurrence (Theorem 2.1). The key of the proof is to show that any compact group extension defined below always preserves d-recurrence for all $d \geq 1$. Let $\mathbf{Y} = (Y, \nu, S)$ be a measure preserving system with a σ-finite measure ν and G a compact group with a right Haar measure ρ with $\rho(G) = 1$. Any measurable map $\phi: Y \to G$ defines a skew product transformation $T: Y \times G \to Y \times G$ by

$$T(y, g) = (Sy, g \cdot \phi(y)).$$

Then the system $\mathbf{X} = (Y \times G, \nu \times \rho, T)$ or the natural projection $\alpha: Y \times G \to Y$ is said to be a **compact group extension** or G-extension of $\mathbf{Y}$, and it is a measure preserving extension. We will note that in the case of non-measure preserving extensions such a lifting fails (Remark 2.1).

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2. Isometric extensions

In this section we will show that all isometric extensions preserve d-recurrence for all $d \geq 1$ and hence multiple recurrence. Thus any ergodic finite extension preserves such a property. A key for the proof is to show that any compact group extension preserves d-recurrence for all $d \geq 1$.

THEOREM 2.1: *Let $d \geq 1$. If the measure preserving system $\mathbf{Y}$ is d-recurrent, then so is any isometric extension $\mathbf{X}$ of $\mathbf{Y}$. Hence if $\mathbf{Y}$ is multiply recurrent, then so is $\mathbf{X}$.*

Before proving the theorem let us recall the following lemma.

LEMMA 2.1 (van der Waerden theorem [Fr]): *Let $L \geq 1$, $d \geq 1$. Then there exists an integer $N = N(L, d) \geq 1$ such that for any finite partition $\{C_1, C_2, \ldots, C_L\}$ of the set $[1, N]$, there are $q, m \in \mathbf{N}$ and $1 \leq \delta \leq L$ such that*

$$q, q + m, q + 2m, \ldots, q + dm \in C_\delta.$$

Here, $[1, N]$ *means the integer interval* $\{1, 2, \ldots, N\}$.

We note that $1 \leq q \leq N$ and $1 \leq m \leq N/d$.

Proof of Theorem 2.1: Since any factor of a d-recurrent system is d-recurrent too, in order to prove the theorem, it is enough to show that any compact group extension preserves d-recurrence for all $d \geq 1$.

Let G be a compact group with a right Haar probability measure ρ and $\mathbf{X} = (X, \mu, T)$ be a G-extension of $\mathbf{Y} = (Y, \nu, S)$ where $X = Y \times G$ and $\mu = \nu \times \rho$. Let $\varepsilon > 0$ and choose a measurable subset E of X with $\mu(E) > \varepsilon$. Then the set $A = \{y \in Y \colon \mu_y(E) > \varepsilon\}$ has positive measure, where μ_y $(y \in Y)$ stands for the conditional fibre probability measure defined by $\mu_y(E) = \int_G 1_E(y, g) d\rho(g) = \rho(E_y)$, $E_y = \{g \in G \colon (y, g) \in E\}$. We will show that there exists an integer $k \geq 1$ such that

$$\mu(E \cap T^{-k}E \cap \cdots \cap T^{-kd}E) > 0.$$

Take an integer $\ell \geq 1$ and a finite partition $\{A_j\}_{j=1,2,\ldots,\ell}$ of Y and measurable subsets $\{B_j\}_{j=1,2,\ldots,\ell}$ of G such that

$$\mu\Big(E \triangle \bigcup_{j=1}^{\ell} (A_j \times B_j)\Big) < \Big\{\frac{\varepsilon}{4(d+1)}\Big\}^2 \nu(A).$$

Put $E' = \bigcup_{j=1}^{\ell} (A_j \times B_j)$ and $A' = \{y \in A \colon \mu_y(E \triangle E') < \varepsilon/4(d+1)\}$; then we see $\nu(A') > 0$.

First we claim that for each $1 \leq j \leq \ell$, the functions $h \cdot 1_{B_j}, h \in G$ are approximated by a finite number of $L^2(G, \rho)$-functions in fiberwise, where $(h \cdot 1_{B_j})(g) = 1_{B_j}(gh), g \in G$. Since G is compact, the set $\{h \cdot 1_{B_j} \colon h \in G\}$ is a precompact subset of $L^2(G, \rho)$ (Proposition 6.5 [Fr]), namely, there are $f_1^{(j)}, f_2^{(j)}, \ldots, f_{L_j}^{(j)} \in L^2(G, \rho)$, with $1 \leq L_j < \infty$ such that

$$\min_{1 \leq i \leq L_j} \|h \cdot 1_{B_j}(g) - f_i^{(j)}(g)\|_{L^2(G,\rho)} < \frac{\varepsilon}{4(d+1)} \tag{1}$$

for all $h \in G$. Then we can take a measurable map $\theta_j \colon G \to [1, L_j]$, $1 \leq j \leq \ell$, defined by

$$\|h \cdot 1_{B_j} - f_{\theta_j(h)}^{(j)}\|_{L^2(G,\rho)} = \min_{1 \leq i \leq L_j} \|h \cdot 1_{B_j} - f_i^{(j)}\|_{L^2(G,\rho)} \tag{2}$$

for all $h \in G$.

Let $\phi(y, n)$ be the cocycle function $Y \times \mathbb{Z} \to G$ determined by

$$T^n(y, g) = (S^n y, g\phi(y, n)).$$

Now we put

$$f_0 \equiv 0,$$
$$f_{i+\sum_{s=0}^{j-1} L_s}(g) = f_i^{(j)}(g), g \in G, \quad \text{for } 1 \leq i \leq L_j, 1 \leq j \leq \ell$$

and

$$L = \sum_{s=0}^{\ell} L_s \quad \text{where } L_0 = 0.$$

We extend f_i to square integrable functions on X by putting

$$f_i(y,g) = f_i(g), \quad 0 \leq i \leq L,$$

and define a measurable map $k: Y \times \mathbb{Z} \to [0, L]$ by

$$k(y,n) = \begin{cases} \theta_j(\phi(y,n)) + \sum_{s=0}^{j-1} L_s & \text{if } S^n y \in A_j,\ 1 \leq j \leq \ell, \\ 0 & \text{otherwise.} \end{cases}$$

It then follows from (1) and (2) that

$$\begin{aligned} &\|T^n 1_{A_j \times B_j}(y,g) - f_{k(y,n)}(y,g)\|_y \\ &\qquad = \|1_{A_j}(S^n y) 1_{B_j}(g\phi(y,n)) - f^{(j)}_{\theta_j(\phi(y,n))}(g)\|_y \\ &\qquad = \|(\phi(y,n) \cdot 1_{B_j})(g) - f^{(j)}_{\theta_j(\phi(y,n))}(g)\|_y \\ &\qquad < \frac{\varepsilon}{4(d+1)} \end{aligned} \tag{3}$$

for $n \in \mathbb{Z}$ with $S^n y \in A_j$, where $\|\cdot\|_y$ stands for the fiber norm and is defined by

$$\|f(y,g)\|_y^2 = \int_G |f(y,g)|^2 d\rho(g),$$

and

$$T^n f(y,g) = f(T^n(y,g)), \quad f \in L^2(X,\mu).$$

Putting

$$k_p(y,n) = k(y,np), \quad p \geq 1,$$

it follows from (3) that

$$\|T^{np} 1_{E'}(y,g) - f_{k_p(y,n)}(y,g)\|_y < \frac{\varepsilon}{4(d+1)} \tag{4}$$

for $y \in Y, n \in \mathbb{Z}$ and $p \geq 1$.

We also put, for $0 \leq j \leq L$ and $y \in Y$,

$$\mathbb{Z}_j(p,y) = \{n \in \mathbb{Z}: k_p(y,n) = j\}.$$

For the integers L and d take an integer $N = N(L,d)$ satisfying the property in Lemma 2.1. Apply multiple recurrence of $\mathbf{Y}$ to get an integer $p_0 \geq 1$ such that

$$\nu\left(\bigcap_{j=1}^{N} S^{-jp_0}A'\right) > 0.$$

If $y \in \bigcap_{j=1}^{N} S^{-jp_0}A'$ and $1 \leq n \leq N$, then $S^{np_0}y \in A'$ and hence $S^{np_0}y \in A_j$ for some $1 \leq j \leq \ell$. By letting $i = k_{p_0}(y,n) = \theta_j(\phi(y,np_0)) + \sum_{s=0}^{j-1} L_s \in [1,L]$, we have that $n \in \mathbb{Z}_i(p_0,y)$. Thus we have a partition

$$[1,N] = \bigcup_{j=1}^{L} \mathbb{Z}_j(p_0,y) \cap [1,N]$$

for $y \in \bigcap_{j=1}^{N} S^{-jp_0}A'$.

Apply the van der Waerden Theorem (Lemma 2.1) for the partition above. Then there are integers $1 \leq q = q(p_0,y) \leq N$, $1 \leq m = m(p_0,y) \leq N/d$ and $1 \leq \delta = \delta(p_0,y) \leq L$ such that

$$q, q+m, q+2m, \ldots, q+dm \in \mathbb{Z}_\delta(p_0,y) \cap [1,N]$$

for $\forall y \in \bigcap_{j=1}^{N} S^{-jp_0}A'$. We note that the functions $y \in \bigcap_{j=1}^{N} S^{-jp_0}A' \to q(p_0,y)$ and $m(p_0,y)$ are measurably chosen. That is,

$$k_{p_0}(y, q(p_0,y) + im(p_0,y)) = \delta(p_0,y) \tag{5}$$

for all $0 \leq i \leq d$. This immediately implies that if we set, for $0 \leq i \leq d$,

$$F_{p_0,i}(y,g) = f_{k_{p_0}(y,q(p_0,y)+im(p_0,y))}(y,g), \quad (y,g) \in \bigcap_{j=1}^{N} S^{-jp_0}A' \times G,$$

then these functions $F_{p_0,i}$ do not depend on i, that is,

$$\begin{aligned} F_{p_0,i}(y,g) &= F_{p_0,0}(y,g) \\ &= f_{k_{p_0}(y,q(p_0,y))}(y,g). \end{aligned}$$

Choose $y \in \bigcap_{j=1}^{N} S^{-jp_0}A'$ and put, for $0 \leq i \leq d$,

$$y_i = S^{(q(p_0,y)+im(p_0,y))p_0}y.$$

It then follows from (4) that for all $0 \le i \le d$,

$$\begin{aligned}
&\mu_{y_0}((T^{-im(p_0,y)p_0}E')\triangle E')\\
&=\int_G |T^{im(p_0,y)p_0}1_{E'}(y,g)-1_{E'}(y,g)|d\mu_{y_0}(g)\\
&=\int_G |T^{(q(p_0,y)+im(p_0,y))p_0}1_{E'}(y,g)-T^{q(p_0,y)p_0}1_{E'}(y,g)|d\mu_y(g)\\
&\le\int_G |T^{(q(p_0,y)+im(p_0,y))p_0}1_{E'}(y,g)-F_{p_0,i}(y,g)|d\mu_y(g)\\
&\quad+\int_G |F_{p_0,0}(y,g)-T^{q(p_0,y)p_0}1_{E'}(y,g)|d\mu_y(g)\\
&\le\|T^{(q(p_0,y)+im(p_0,y))p_0}1_{E'}(y,g)-F_{p_0,i}(y,g)\|_y\\
&\quad+\|T^{q(p_0,y)p_0}1_{E'}(y,g)-F_{p_0,0}(y,g)\|_y\\
&<\frac{\varepsilon}{4(d+1)}+\frac{\varepsilon}{4(d+1)}=\frac{\varepsilon}{2(d+1)}.
\end{aligned}$$

We note that $y_i \in A'$ for all $0 \le i \le d$, since $q(p_0,y)+im(p_0,y) \in [1,N]$, $0 \le \forall i \le d$ and $y \in \bigcap_{j=1}^{N} S^{-jp_0}A'$. Therefore

$$\begin{aligned}
\mu_{y_0}((T^{-im(p_0,y)p_0}E)\triangle E) \le&\mu_{y_0}(T^{-im(p_0,y)p_0}(E\triangle E'))\\
&+\mu_{y_0}((T^{-im(p_0,y)p_0}E')\triangle E')+\mu_{y_0}(E'\triangle E)\\
=&\mu_{y_i}(E\triangle E')\\
&+\mu_{y_0}((T^{-im(p_0,y)p_0}E')\triangle E')+\mu_{y_0}(E'\triangle E)\\
<&\frac{\varepsilon}{4(d+1)}+\frac{\varepsilon}{2(d+1)}+\frac{\varepsilon}{4(d+1)}=\frac{\varepsilon}{d+1}.
\end{aligned}$$

Thus we have

$$\mu_{y_0}\left(\left(\bigcap_{i=0}^{d}T^{-im(p_0,y)p_0}E\right)\triangle E\right)<\varepsilon,$$

and hence

$$\mu_{y_0}\left(\bigcap_{i=0}^{d}T^{-im(p_0,y)p_0}E\right)>\mu_{y_0}(E)-\varepsilon \tag{6}$$

for $y \in \bigcap_{j=1}^{N} S^{-jp_0}A'$.

Choose a smaller subset $A_0 \subset \bigcap_{j=1}^{N} S^{-jp_0}A'$ of positive measure such that the functions $q(p_0,y)$ and $m(p_0,y)$ are constant, q_0 and m_0 respectively, that is,

$$\begin{aligned}
q(p_0,y)&=q_0, \quad y\in A_0,\\
m(p_0,y)&=m_0, \quad y\in A_0.
\end{aligned}$$

We note that $1 \leq q_0 \leq N$ and $S^{q_0 p_0} y \in A' \subset A$ for $y \in A_0$, and hence that $\mu_{S^{q_0 p_0} y}(E) > \varepsilon$. It then follows from (6) that

$$\begin{aligned}
&\mu(E \cap T^{-m_0 p_0} E \cap \cdots \cap T^{-d m_0 p_0} E) \\
&\qquad = \int_Y \mu_y(E \cap T^{-m_0 p_0} E \cap \cdots \cap T^{-d m_0 p_0} E) d\nu(y) \\
&\qquad = \int_Y \mu_{S^{q_0 p_0} y}(E \cap T^{-m_0 p_0} E \cap \cdots \cap T^{-d m_0 p_0} E) d\nu(y) \\
&\qquad \geq \int_{A_0} \mu_{S^{q_0 p_0} y}(E \cap T^{-m_0 p_0} E \cap \cdots \cap T^{-d m_0 p_0} E) d\nu(y) \\
&\qquad > \int_{A_0} (\mu_{y_0}(E) - \varepsilon) d\nu(y) > 0.
\end{aligned}$$

Thus we complete the proof. ∎

Remark 2.1: We know that the symmetric random walk on $\mathbb{Z}$ is a group $\mathbb{Z}$-extension of the (multiply recurrent) 2-shift, and that it is 2-recurrent but not 3-recurrent by using the local limit theorem (page 195 [A]), [A] (page 196) and the Aaronson–Nakada's criterion for Markov shifts (page 292 Th. 1.1 [A-N]). This means that some non-compact group extension, which is a ∞-factor map though, preserves neither multiple recurrence nor 3-recurrence.

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